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Hyperbolic heat conduction and thermal resonances in a cylindrical solid carrying a steady-periodic electric field

A. BARLETTA and E. ZANCHINI

Istituto di Fisica Tecnica, Facoltà di Ingegneria, Università di Bologna, Viale Risorgimento 2, 40136 Bologna, Italy

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Abstract—Hyperbolic heat conduction in an infinitely long cylindrical solid with internal heat generation produced by Joule effect is considered. The power generated per unit volume is non-uniform and steady-periodic. The surface of the cylinder is assumed to exchange heat by convection with an external fluid. The temperature field within the cylinder is determined analytically in a steady-periodic regime. For a fixed material and for a fixed radius of the cylinder, the dependence of the amplitude of thermal waves on the frequency of the electric current is studied and it is proved that thermal resonances occur.

INTRODUCTION

It is well known that, in non-stationary heat conduction problems, Fourier's law implies an infinite speed for the propagation of thermal signals. The hyperbolic heat conduction theory is based on a constitutive equation for the heat flux vector which coincides with Fourier's law for stationary problems, and which predicts a finite speed for the propagation of thermal signals for non-stationary problems. A clear and exhaustive review of this theory has been recently proposed by Özisik and Tzou [1]. In the constitutive equation for the heat flux vector, which is proposed in hyperbolic heat conduction theory, a new physical quantity is introduced: the *relaxation time* τ. As is widely explained in refs. [1, 2], the relaxation time represents the time-lag between the temperature gradient and the resulting heat flux vector. Generally, such a time-lag can be neglected for slowly varying temperature fields, but becomes significant in non-stationary problems where local temperature variations take place in time intervals with durations comparable to τ. The relaxation time τ is directly related to the propagation speed of thermal waves: the latter is given by the square root of the ratio α/τ, where α is the thermal diffusivity of the material. According to the hyperbolic heat conduction theory, the relaxation time τ is a thermodynamic property of the material. However, apart from experiments at very low temperatures on liquid helium II [3], measurements of the relaxation time or of the propagation speed of thermal waves, especially at room or at elevated temperatures, are very rare in the literature. Recently, measurements on non-homogeneous materials at about 20°C have been performed by Kaminski [4]. The materials considered by Kaminski are H acid, $NaHCO_3$, sand, glass ballotini and an ion exchanger. The experimental values of τ determined in ref. [4] range from 10.9 s for glass ballotini to 53.7 s for the ion exchanger. On the contrary, only theoretical guesses of the value of τ for homogeneous solid materials at room or high temperatures are available in the literature. For instance, it is usually retained that metals have values of τ about 10^{-14}–10^{-11} s [1, 2]. Since rigorous tables of relaxation times or of thermal wave speeds for engineering materials are not yet available, exact solutions of the hyperbolic heat conduction equation can be very useful in the design of new experimental setups for measurements of thermal waves propagation. Many exact solutions of the hyperbolic heat conduction equation have been found in the literature (a wide list of references can be found in ref. [1]). In our opinion, one of the most interesting features of hyperbolic heat conduction is the resonance phenomenon which has been recognized in thermal wave propagation by Tzou [5, 6].

In refs. [5, 6], the hyperbolic heat conduction equation is solved for an infinitely wide slab of solid material with an internal heat source. In particular, the author assumes that, within the slab, there exists a periodic and non-uniform power generated per unit volume such that its time-average is zero. Since the time-average of the power generated per unit volume is zero, heat is alternately generated and subtracted within the slab. The author prescribes either a zero wall heat flux [5] or a fixed temperature [6] on the two faces of the slab, and determines the amplitude of thermal waves in steady-periodic regime. In refs. [5, 6], it is shown that there exist values of the frequency of the heat source which determine a resonant behaviour of the thermal waves. However, the results reported in refs. [5, 6] do not seem to have a direct physical applicability, because the heat source con-

NOMENCLATURE

Bi	$= hr_0/k$, Biot number
$\mathbf{E}$	electric field [V m^{-1}]
E	axial component of $\mathbf{E}$ [V m^{-1}]
h	convection heat transfer coefficient [W m^{-2} K^{-1}]
$\mathbf{H}$	magnetic field [A m^{-1}]
i	$= \sqrt{-1}$, imaginary unit
Im	imaginary part of a complex number
J_n	Bessel function of first kind and order n
$\mathbf{J}$	electric current density [A m^{-2}]
k	thermal conductivity [W m^{-1} K^{-1}]
k_1, k_2	dimensionless constants introduced in equation (35)
$\mathbf{q}$	heat flux vector [W m^{-2}]
q_g	power generated per unit volume [W m^{-3}]
$\tilde{q}_g$	source term in the hyperbolic heat conduction equation [W m^{-3}]
q_r	radial component of the heat flux vector [W m^{-2}]
$\dot{Q}$	time-averaged power generated per unit length [W m^{-1}]
r	radial coordinate [m]
r_0	radius of the cylinder [m]
Re	real part of a complex number
s	$= \omega t$, dimensionless time
t	time [s]
T	temperature [K]
T_f	fluid temperature outside the boundary layer [K]
T_w	surface temperature [K]
$\overline{T_w}$	time-averaged surface temperature [K]
$\mathcal{T}$	$= \Gamma/\Omega^2$, dimensionless parameter
u	function of η, Γ, Λ, Ω defined in equation (39)
v	function of η, Γ, Λ, Ω defined in equation (40)
w	Wronskian
Y_n	Bessel function of second kind and order n
$\equiv$	equal by definition
$\| \|$	modulus of a complex number.

Greek symbols

α	thermal diffusivity [m^2 s^{-1}]
γ	time independent part of $\mathbf{E}$, defined in equation (6) [V m^{-1}]
Γ	$= \omega\,\tau$, dimensionless parameter
ε	electric permittivity [F m^{-1}]
η	$= r/r_0$, dimensionless radius
η', η''	integration variables
ϑ	dimensionless temperature
$\bar{\vartheta}$, ϑ_1, ϑ_2	dimensionless functions defined in equation (22)
Θ	$= \Lambda^2/\Omega^2$, dimensionless parameter
Λ	$= (2\omega/\alpha)^{1/2} r_0$, dimensionless parameter
μ	magnetic permeability [V s A^{-1} m^{-1}]
σ	electric conductivity [A V^{-1} m^{-1}]
τ	relaxation time in the constitutive equation (10) [s]
ψ	$= \vartheta_1 + \mathrm{i}\vartheta_2$, dimensionless function
ω	angular frequency [rad s^{-1}]
Ω	$= (\omega\mu\sigma)^{1/2} r_0$, dimensionless parameter.

sidered in the analysis can hardly be reproduced experimentally.

The aim of this paper is to determine analytically the steady-periodic temperature field which solves the hyperbolic heat conduction equation for an infinitely long cylindrical solid carrying an alternating electric field with a given frequency. The external surface of the cylinder will be supposed to exchange heat by convection with a surrounding fluid. It will be shown that a resonance phenomenon for thermal waves occurs in this system.

Indeed, the presence of an alternating electric field within the cylinder determines, on account of Ohm's law, an alternating electric current and a consequent heat generation by Joule effect. Let us recall that, when the current is alternating, the power generated per unit volume by Joule effect is not uniformly distributed because of the *skin effect*, which can be described as follows. As the frequency of the electric current increases, the electric current density within the solid tends to assume significant values only in the neighbourhood of the external surface (skin) and tends to be negligible in the interior [7, 8]. As a consequence, the power generated per unit volume by Joule effect has a non-uniform distribution.

In the literature, many papers deal with Fourier's (i.e. parabolic) heat conduction within a solid crossed by an alternating current [9–16]. However, no such study in the framework of hyperbolic heat conduction theory is yet available.

MATHEMATICAL MODEL

In this section, the evaluation of the steady-periodic power per unit volume generated by Joule effect within an infinitely long cylindrical solid crossed by an alternating electric current, presented in ref. [14], is outlined. Then, the hyperbolic heat conduction equation for the problem under examination is presented.

Let us consider an infinitely long cylindrical solid with radius r_0. It will be assumed that the thermal and electric properties of the solid are independent of temperature, so that they can be treated as constants. A parallel electric field $\mathbf{E}$, which depends only on r

and t and is directed axially, is present within the solid. This field depends periodically on time, with an angular frequency ω. The macroscopic charge density distribution in the solid is zero. It will be assumed that the angular frequency ω of the electric field oscillations satisfies the condition $\omega\varepsilon \ll \sigma$, so that the electric displacement current can be neglected. This assumption is called *quasi-stationary approximation* of the electromagnetic field equations [7] and is satisfied even at very high frequencies. In fact, for a material with $\sigma = 10^7$ A V^{-1} m^{-1} and $\varepsilon = 10^{-11}$ F m^{-1}, the condition $\omega\varepsilon \ll \sigma$ holds if $\omega \ll 10^{18}$ rad s^{-1}.

In the quasi-stationary approximation, Maxwell's equations can be written as [7]

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} \tag{1}$$

$$\nabla \times \mathbf{H} = \mathbf{J} \tag{2}$$

$$\nabla \cdot \mathbf{H} = 0 \tag{3}$$

$$\nabla \cdot \mathbf{E} = 0. \tag{4}$$

Moreover, Ohm's law is supposed to hold, i.e.

$$\mathbf{J} = \sigma \mathbf{E}. \tag{5}$$

In ref. [14], it is proved that the axial component of $\mathbf{E}$ can be expressed as

$$E(r,t) = \gamma(r)\,\mathrm{e}^{-\mathrm{i}\omega t} \tag{6}$$

where function $\gamma(r)$ is given by

$$\gamma(r) = \left[\frac{\omega\mu\dot{Q}}{\pi\Omega^2 f(\Omega)}\right]^{1/2} J_0(\sqrt{\mathrm{i}}\,\Omega\eta) \tag{7}$$

and $f(\Omega)$ is defined as

$$f(\Omega) \equiv \int_0^1 \|J_0(\sqrt{\mathrm{i}}\,\Omega\eta)\|^2 \eta\,\mathrm{d}\eta. \tag{8}$$

In equations (7) and (8), the dimensionless parameter $\Omega = (\omega\mu\sigma)^{1/2} r_0$ and the dimensionless radial coordinate $\eta = r/r_0$ have been employed. The power per unit volume generated within the solid by Joule effect is $q_g = \mathrm{Re}(\mathbf{J}\cdot\mathbf{E})$ [17], so that equations (5)–(8) yield

$$\begin{aligned} q_g(r,t) = \frac{\dot{Q}}{2\pi r_0^2 f(\Omega)} [&\|J_0(\sqrt{\mathrm{i}}\,\Omega\eta)\|^2 \\ &+ \mathrm{Re}(J_0(\sqrt{\mathrm{i}}\,\Omega\eta)^2)\cos(2\omega t) \\ &+ \mathrm{Im}(J_0(\sqrt{\mathrm{i}}\,\Omega\eta)^2)\sin(2\omega t)]. \end{aligned} \tag{9}$$

On account of equation (9), it is easily proved that $q_g(r,t)$ oscillates with twice the frequency of the electric field oscillations and with an amplitude equal to its mean value.

In hyperbolic heat conduction, the heat flux vector $\mathbf{q}$ does not obey Fourier's law, $\mathbf{q} = -k\nabla T$, but the constitutive equation [1]

$$\mathbf{q} + \tau \frac{\partial \mathbf{q}}{\partial t} = -k\nabla T. \tag{10}$$

The local energy balance can be expressed as

$$-\nabla \cdot \mathbf{q} + q_g(r,t) = \frac{k}{\alpha}\frac{\partial T}{\partial t}. \tag{11}$$

By employing equations (10) and (11), it is easily proved that the temperature field must obey the hyperbolic heat conduction equation [1]

$$k\nabla^2 T + \tilde{q}_g(r,t) = \frac{k}{\alpha}\left[\frac{\partial T}{\partial t} + \tau \frac{\partial^2 T}{\partial t^2}\right] \tag{12}$$

where the source term $\tilde{q}_g(r,t)$ is given by

$$\tilde{q}_g(r,t) = q_g(r,t) + \tau \frac{\partial q_g(r,t)}{\partial t}. \tag{13}$$

As a consequence of equations (9) and (13), the source term $\tilde{q}_g(r,t)$ can be expressed as

$$\begin{aligned} \tilde{q}_g(r,t) = \frac{\dot{Q}}{2\pi r_0^2 f(\Omega)} \{&\|J_0(\sqrt{\mathrm{i}}\,\Omega\eta)\|^2 \\ &+ [\mathrm{Re}(J_0(\sqrt{\mathrm{i}}\,\Omega\eta)^2) + 2\omega\tau\,\mathrm{Im}(J_0(\sqrt{\mathrm{i}}\,\Omega\eta)^2)]\cos(2\omega t) \\ &+ [\mathrm{Im}(J_0(\sqrt{\mathrm{i}}\,\Omega\eta)^2) - 2\omega\tau\,\mathrm{Re}(J_0(\sqrt{\mathrm{i}}\,\Omega\eta)^2)]\sin(2\omega t)\}. \end{aligned} \tag{14}$$

DIMENSIONLESS FORM OF THE HYPERBOLIC HEAT CONDUCTION EQUATION

In this section, the hyperbolic heat conduction equation is written in a dimensionless form. Then, under the hypothesis that the heat conduction is steady-periodic, this equation is transformed into a system of three ordinary differential equations.

Let convection be present at the surface of the cylinder with an external fluid which has temperature T_f outside the boundary layer. Then, the boundary condition at $r = r_0$ can be expressed as

$$q_r(r_0,t) = h[T_w(t) - T_f] \tag{15}$$

so that, on account of equation (10), the temperature field at $r = r_0$ must fulfil the condition

$$-k\left.\frac{\partial T}{\partial r}\right|_{r=r_0} = h\left[T_w(t) + \tau \frac{\mathrm{d}T_w(t)}{\mathrm{d}t} - T_f\right]. \tag{16}$$

The time-averaged power generated within the cylinder per unit length, $\dot{Q}$, is related to $\overline{T_w}$ by

$$\dot{Q} = 2\pi r_0 h(\overline{T_w} - T_f). \tag{17}$$

On account of equation (17), if a dimensionless temperature

$$\vartheta = \frac{T - T_f}{\overline{T_w} - T_f} \tag{18}$$

is defined and the dimensionless variables η and $s = \omega t$ are introduced together with the Biot number $Bi = hr_0/k$, then equation (12) can be rewritten as

$$\frac{1}{\eta}\frac{\partial}{\partial\eta}\left(\eta\frac{\partial\vartheta}{\partial\eta}\right)+Bi\frac{2\pi r_0^2}{Q}\tilde{q}_g=\frac{\omega r_0^2}{\alpha}\left[\frac{\partial\vartheta}{\partial s}+\omega\tau\frac{\partial^2\vartheta}{\partial s^2}\right]. \quad (19)$$

By employing equation (14), the dimensionless parameter $\Gamma=\omega\tau$ and the non-negative dimensionless parameter Λ such that $\Lambda^2/2=\omega r_0^2/\alpha$, equation (19) can be rewritten as

$$\frac{1}{\eta}\frac{\partial}{\partial\eta}\left(\eta\frac{\partial\vartheta}{\partial\eta}\right)+\frac{Bi}{f(\Omega)}\{\|J_0(\sqrt{i}\,\Omega\eta)\|^2$$
$$+[\mathrm{Re}(J_0(\sqrt{i}\,\Omega\eta)^2)+2\Gamma\,\mathrm{Im}(J_0(\sqrt{i}\,\Omega\eta)^2)]\cos(2s)$$
$$+[\mathrm{Im}(J_0(\sqrt{i}\,\Omega\eta)^2)-2\Gamma\,\mathrm{Re}(J_0(\sqrt{i}\,\Omega\eta)^2)]\sin(2s)\}$$
$$=\frac{\Lambda^2}{2}\left(\frac{\partial\vartheta}{\partial s}+\Gamma\frac{\partial^2\vartheta}{\partial s^2}\right). \quad (20)$$

On account of equation (18) and of the definition of Bi, equation (16) yields

$$\left.\frac{\partial\vartheta}{\partial\eta}\right|_{\eta=1}+Bi\left[\vartheta(1,s)+\Gamma\frac{\partial\vartheta(1,s)}{\partial s}\right]=0. \quad (21)$$

As a consequence of equations (20) and (21), ϑ is a function of η and s which depends parametrically on Bi, Γ, Λ and Ω. After a sufficiently long time, the heat conduction becomes steady-periodic and the solution of equations (20) and (21) has the form

$$\vartheta(\eta,s)=\bar{\vartheta}(\eta)+\vartheta_1(\eta)\cos(2s)+\vartheta_2(\eta)\sin(2s). \quad (22)$$

While $\bar{\vartheta}$ represents the average of ϑ with respect to s in the interval $[0,\pi]$, ϑ_1 and ϑ_2 have the following physical meaning. The complex valued function $\psi\equiv\vartheta_1+i\vartheta_2$ is such that its modulus and its argument represent, respectively, the amplitude and the phase of the oscillations of ϑ around its mean value $\bar{\vartheta}$.

By substituting equation (22) into equation (20), one obtains

$$\frac{1}{\eta}\frac{d}{d\eta}\left(\eta\frac{d\bar{\vartheta}}{d\eta}\right)+\frac{Bi}{f(\Omega)}\|J_0(\sqrt{i}\,\Omega\eta)\|^2$$
$$+\left\{\frac{1}{\eta}\frac{d}{d\eta}\left(\eta\frac{d\vartheta_1}{d\eta}\right)+\frac{Bi}{f(\Omega)}[\mathrm{Re}(J_0(\sqrt{i}\,\Omega\eta)^2)\right.$$
$$+2\Gamma\,\mathrm{Im}(J_0(\sqrt{i}\,\Omega\eta)^2)]-\Lambda^2\vartheta_2$$
$$\left.+2\Lambda^2\Gamma\vartheta_1\right\}\cos(2s)+\left\{\frac{1}{\eta}\frac{d}{d\eta}\left(\eta\frac{d\vartheta_2}{d\eta}\right)\right.$$
$$+\frac{Bi}{f(\Omega)}[\mathrm{Im}(J_0(\sqrt{i}\,\Omega\eta)^2)-2\Gamma\,\mathrm{Re}(J_0(\sqrt{i}\,\Omega\eta)^2)]$$
$$\left.+\Lambda^2\vartheta_1+2\Lambda^2\Gamma\vartheta_2\right\}\sin(2s)=0. \quad (23)$$

The integration with respect to s of both sides of equation (23) in the interval $[0,\pi]$ yields

$$\frac{1}{\eta}\frac{d}{d\eta}\left(\eta\frac{d\bar{\vartheta}}{d\eta}\right)+\frac{Bi}{f(\Omega)}\|J_0(\sqrt{i}\,\Omega\eta)\|^2=0. \quad (24)$$

The multiplication by $\cos(2s)$ of both sides of equation (23) and the integration with respect to s in the interval $[0,\pi]$ yields

$$\frac{1}{\eta}\frac{d}{d\eta}\left(\eta\frac{d\vartheta_1}{d\eta}\right)+\frac{Bi}{f(\Omega)}[\mathrm{Re}(J_0(\sqrt{i}\,\Omega\eta)^2)$$
$$+2\Gamma\,\mathrm{Im}(J_0\sqrt{i}\,\Omega\eta)^2)]-\Lambda^2\vartheta_2+2\Lambda^2\Gamma\vartheta_1=0. \quad (25)$$

The multiplication by $\sin(2s)$ of both sides of equation (23) and the integration with respect to s in the interval $[0,\pi]$ yields

$$\frac{1}{\eta}\frac{d}{d\eta}\left(\eta\frac{d\vartheta_2}{d\eta}\right)+\frac{Bi}{f(\Omega)}[\mathrm{Im}(J_0(\sqrt{i}\,\Omega\eta)^2)$$
$$-2\Gamma\,\mathrm{Re}(J_0(\sqrt{i}\Omega\eta)^2)]+\Lambda^2\vartheta_1+2\Lambda^2\Gamma\vartheta_2=0. \quad (26)$$

Therefore, the dimensionless Fourier equation has been transformed into a system of three coupled differential equations in the variable η, namely equations (24)–(26). Moreover, by substituting equation (22) into equation (21), one obtains

$$\left.\frac{d\bar{\vartheta}}{d\eta}\right|_{\eta=1}+Bi\bar{\vartheta}(1)+\left\{\left.\frac{d\vartheta_1}{d\eta}\right|_{\eta=1}\right.$$
$$\left.+Bi[\vartheta_1(1)+2\Gamma\vartheta_2(1)]\right\}\cos(2s)$$
$$+\left\{\left.\frac{d\vartheta_2}{d\eta}\right|_{\eta=1}+Bi[\vartheta_2(1)-2\Gamma\vartheta_1(1)]\right\}\sin(2s)=0. \quad (27)$$

By employing the same method which has allowed us to split equation (23) into equations (24)–(26), equation (27) can be transformed into the following set of equations:

$$\left.\frac{d\bar{\vartheta}}{d\eta}\right|_{\eta=1}+Bi\bar{\vartheta}(1)=0 \quad (28)$$

$$\left.\frac{d\vartheta_1}{d\eta}\right|_{\eta=1}+Bi[\vartheta_1(1)+2\Gamma\vartheta_2(1)]=0 \quad (29)$$

$$\left.\frac{d\vartheta_2}{d\eta}\right|_{\eta=1}+Bi[\vartheta_2(1)-2\Gamma\vartheta_1(1)]=0. \quad (30)$$

Equations (24) and (28), which determine the time-averaged distribution of dimensionless temperature, are the same as those that one would obtain by employing Fourier's equation, instead of the hyperbolic heat conduction equation. In other words, equations (24) and (28) are not affected by the relaxation time τ, so that the time-averaged distribution of dimensionless temperature is the same for every value of τ. In ref. [14], an expression of the time-averaged distribution of dimensionless temperature $\bar{\vartheta}$ has been determined analytically in the case of Fourier's heat conduction, i.e. for $\tau=0$, and is given by

$$\bar{\vartheta}(\eta)=1+Bi\,g(\eta,\Omega) \quad (31)$$

where $g(\eta,\Omega)$ is defined as

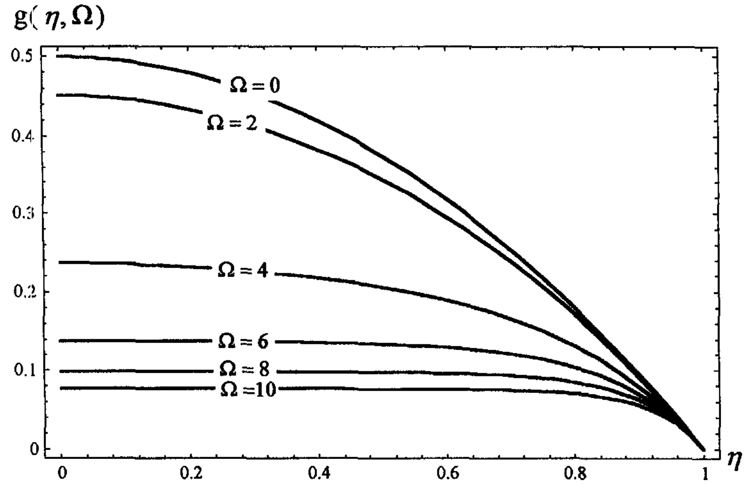

Fig. 1. Plots of $g(\eta, \Omega)$ vs η for various values of Ω.

$$g(\eta,\Omega) \equiv \frac{1}{f(\Omega)} \int_{\eta}^{1} \frac{1}{\eta''} \left(\int_{0}^{\eta''} \eta' \| J_0(\sqrt{\mathrm{i}}\,\Omega\eta) \|^2 \, \mathrm{d}\eta' \right) \mathrm{d}\eta''. \tag{32}$$

Plots of $g(\eta, \Omega)$ for various values of Ω are reported in Fig. 1. These plots show how function $g(\eta, \Omega)$ tends to become uniform as Ω increases. Indeed, in the limit $\Omega \to +\infty$, the time-averaged dimensionless temperature $\bar{\vartheta}(\eta)$ equals 1 for any value of η.

THERMAL WAVES

In this section, equations (25) and (26) are solved analytically with the boundary conditions expressed by equations (29) and (30).

Equations (25) and (26) are coupled. However, if one employs the complex valued function ψ, equations (25) and (26) collapse into a unique complex differential equation, namely

$$\frac{1}{\eta} \frac{\mathrm{d}}{\mathrm{d}\eta} \left(\eta \frac{\mathrm{d}\psi}{\mathrm{d}\eta} \right) + \Lambda^2 (2\Gamma + \mathrm{i}) \psi = \mathrm{i} \frac{Bi(2\Gamma + \mathrm{i})}{f(\Omega)} J_0(\sqrt{\mathrm{i}}\,\Omega\eta)^2 \tag{33}$$

while equations (29) and (30) collapse into a unique boundary condition, namely

$$\left. \frac{\mathrm{d}\psi}{\mathrm{d}\eta} \right|_{\eta=1} - \mathrm{i} Bi (2\Gamma + \mathrm{i}) \psi(1) = 0. \tag{34}$$

By employing the method of variation of parameters presented in ref. [18] and the properties of Bessel functions [19], the general solution of the inhomogeneous differential equation (33) can be written in the form

$$\begin{aligned} \psi(\eta) = {} & k_1 J_0(\sqrt{2\Gamma + \mathrm{i}}\,\Lambda\eta) + k_2 Y_0(\sqrt{2\Gamma + \mathrm{i}}\,\Lambda\eta) \\ & + \mathrm{i} \frac{Bi}{f(\Omega)} (2\Gamma + \mathrm{i}) Y_0(\sqrt{2\Gamma + \mathrm{i}}\,\Lambda\eta) \\ & \times \int_0^{\eta} \frac{J_0(\sqrt{2\Gamma + \mathrm{i}}\,\Lambda\eta')}{w(\eta')} J_0(\sqrt{\mathrm{i}}\,\Omega\eta')^2 \, \mathrm{d}\eta' \\ & - \mathrm{i} \frac{Bi}{f(\Omega)} (2\Gamma + \mathrm{i}) J_0(\sqrt{2\Gamma + \mathrm{i}}\,\Lambda\eta) \\ & \times \int_0^{\eta} \frac{Y_0(\sqrt{2\Gamma + \mathrm{i}}\,\Lambda\eta')}{w(\eta')} J_0(\sqrt{\mathrm{i}}\,\Omega\eta')^2 \, \mathrm{d}\eta'. \end{aligned} \tag{35}$$

The Wronskian $w(\eta)$, which appears in equation (35) is given by

$$\begin{aligned} w(\eta) = {} & J_0(\sqrt{2\Gamma + \mathrm{i}}\,\Lambda\eta) \frac{\mathrm{d} Y_0(\sqrt{2\Gamma + \mathrm{i}}\,\Lambda\eta)}{\mathrm{d}\eta} \\ & - Y_0(\sqrt{2\Gamma + \mathrm{i}}\,\Lambda\eta) \frac{\mathrm{d} J_0(\sqrt{2\Gamma + \mathrm{i}}\,\Lambda\eta)}{\mathrm{d}\eta} = \frac{2}{\pi\eta} \end{aligned} \tag{36}$$

where the identity [19]

$$J_0(x) \frac{\mathrm{d} Y_0(x)}{\mathrm{d}x} - Y_0(x) \frac{\mathrm{d} J_0(x)}{\mathrm{d}x} = \frac{2}{\pi x} \tag{37}$$

has been employed.

On account of equation (35), function ψ is regular at $\eta = 0$ if and only if $k_2 = 0$. In this case, equation (35) can be rewritten as

$$\begin{aligned} \psi(\eta) = {} & k_1 J_0(\sqrt{2\Gamma + \mathrm{i}}\,\Lambda\eta) + \mathrm{i} \frac{Bi}{f(\Omega)} (2\Gamma + \mathrm{i}) \\ & [Y_0(\sqrt{2\Gamma + \mathrm{i}}\,\Lambda\eta) u(\eta, \Gamma, \Lambda, \Omega) \\ & - J_0(\sqrt{2\Gamma + \mathrm{i}}\,\Lambda\eta) v(\eta, \Gamma, \Lambda, \Omega)] \end{aligned} \tag{38}$$

where functions $u(\eta, \Gamma, \Lambda, \Omega)$ and $v(\eta, \Gamma, \Lambda, \Omega)$ are defined as follows:

$$u(\eta, \Gamma, \Lambda, \Omega) \equiv \frac{\pi}{2}\int_0^\eta J_0(\sqrt{2\Gamma+\mathrm{i}}\,\Lambda\eta')J_0(\sqrt{\mathrm{i}}\,\Omega\eta')^2\eta'\,\mathrm{d}\eta' \quad (39)$$

$$v(\eta, \Gamma, \Lambda, \Omega) \equiv \frac{\pi}{2}\int_0^\eta Y_0(\sqrt{2\Gamma+\mathrm{i}}\,\Lambda\eta')J_0(\sqrt{\mathrm{i}}\,\Omega\eta')^2\eta'\,\mathrm{d}\eta'. \quad (40)$$

On account of the properties of Bessel functions, the substitution of equation (38) into equation (34) yields

$$k_1 = \mathrm{i}\frac{Bi}{f(\Omega)}(2\Gamma+\mathrm{i})\{v(1,\Gamma,\Lambda,\Omega)-u(1,\Gamma,\Lambda,\Omega)$$
$$\times[Bi\sqrt{2\Gamma+\mathrm{i}}\,Y_0(\sqrt{2\Gamma+\mathrm{i}}\,\Lambda\eta)-\mathrm{i}\Lambda Y_1(\sqrt{2\Gamma+\mathrm{i}}\,\Lambda\eta)]$$
$$\times[Bi\sqrt{2\Gamma+\mathrm{i}}\,J_0(\sqrt{2\Gamma+\mathrm{i}}\,\Lambda\eta)$$
$$-\mathrm{i}\Lambda J_1(\sqrt{2\Gamma+\mathrm{i}}\,\Lambda\eta)]^{-1}\}. \quad (41)$$

Equations (38)–(41) determine both the modulus and the argument of ψ, i.e. the amplitude and the phase of the oscillations of ϑ around its mean value $\bar{\vartheta}$.

THE ONSET OF THERMAL RESONANCES

In this section, equations (38)–(41) are employed to analyse how the amplitude of the dimensionless temperature oscillations at the surface of the cylinder depends on the dimensionless parameters Ω and Γ/Ω^2.

Plots of the modulus of ψ as a function of Ω, for $\eta = 1$ and $Bi = 0.1$ are reported in Figs. 2–6. Since both Γ and Λ depend on the angular frequency ω, in Figs. 2–6 a prescribed value is assigned to the dimensionless parameters

$$\mathscr{T} = \frac{\Gamma}{\Omega^2} = \frac{\tau}{\mu\sigma r_0^2} \quad \text{and}\ \Theta = \frac{\Lambda^2}{\Omega^2} = \frac{2}{\alpha\mu\sigma}. \quad (42)$$

In Fig. 2, the modulus of ψ, i.e. the amplitude of the dimensionless temperature oscillations is plotted for $\mathscr{T} = 0$ and $\Theta = 100$. This plot represents the behaviour of the amplitude in the limit of Fourier's heat conduction, i.e. for $\tau = 0$. In particular, Fig. 2 shows that the amplitude of the temperature oscillations is a decreasing function of Ω and becomes practically negligible for $\Omega \gtrsim 0.2$. The effects of a finite relaxation time τ are illustrated in Figs. 3–6. These figures correspond to $\Theta = 100$ and $\mathscr{T}$ equal to 10^3, 10^4, 10^5 and 10^6, respectively.

Figures 3–6 show that, unlike Fourier's heat conduction, hyperbolic heat conduction produces thermal resonances which become more and more significant as materials with increasing relaxation times are considered. In Figs. 2–6, it is shown that, for $\eta = 1$, the modulus of ψ tends to 1 when $\Omega \to 0$. Therefore, on account of equations (31) and (32), when $\Omega \to 0$ the amplitude of the dimensionless temperature oscillations at $\eta = 1$ becomes equal to the time-averaged value of the dimensionless temperature. As Ω increases, the amplitude has a behaviour which strongly depends on $\mathscr{T}$.

In Fig. 3, the amplitude of the dimensionless temperature oscillations initially decreases with Ω, but for $\Omega \gtrsim 0.092$ the amplitude begins to increase and reaches a maximum for $\Omega \cong 0.104$; this value of Ω determines the first resonance. A second resonance occurs for $\Omega \cong 0.136$.

In Fig. 4, the amplitude initially increases with Ω

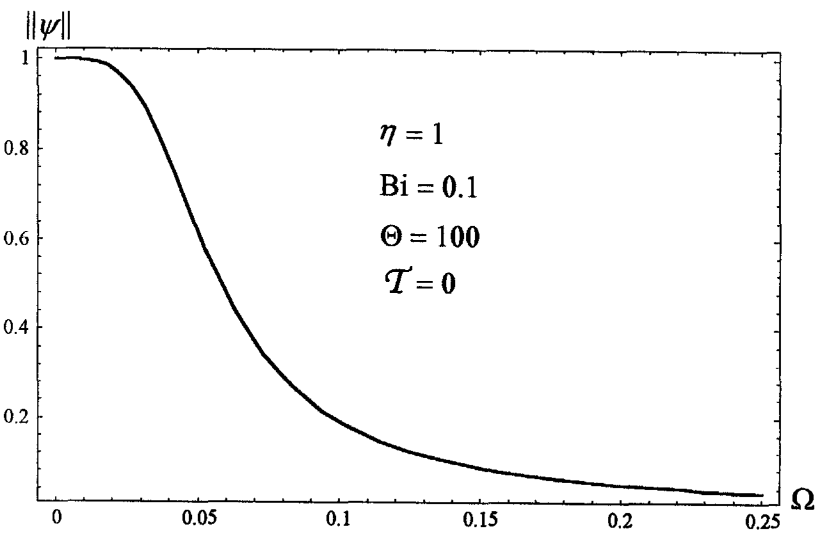

Fig. 2. Plot of the amplitude $\|\psi\|$ as a function of Ω for $\mathscr{T} = 0$.

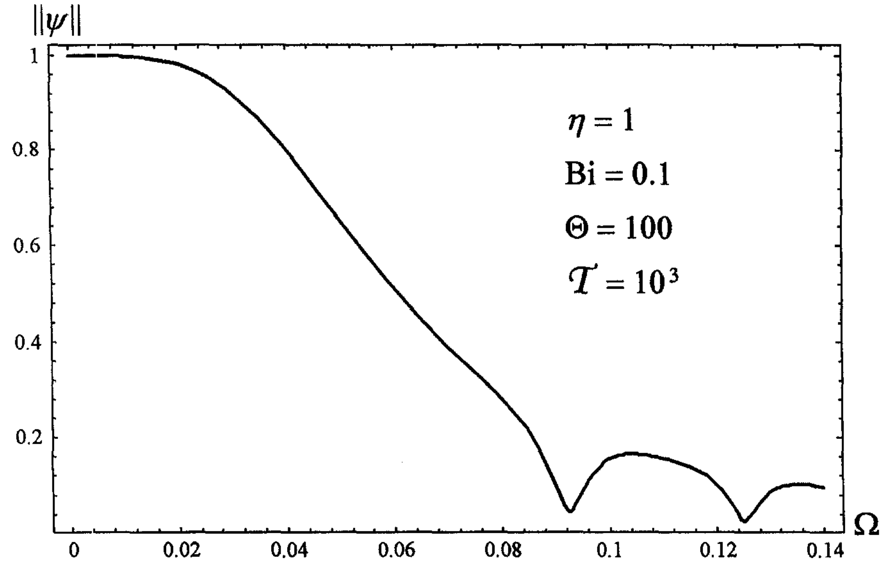

Fig. 3. Plot of the amplitude $\|\psi\|$ as a function of Ω for $\mathcal{T} = 10^3$.

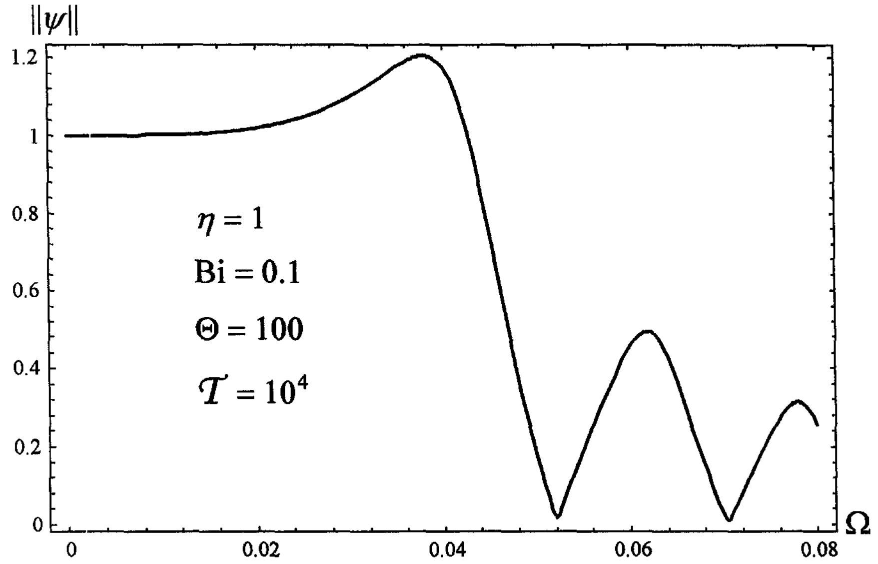

Fig. 4. Plot of the amplitude $\|\psi\|$ as a function of Ω for $\mathcal{T} = 10^4$.

and reaches a maximum for $\Omega \cong 0.038$. Other resonances occur for $\Omega \cong 0.062$, $\Omega \cong 0.078$, etc.

In Fig. 5, the amplitude initially increases with Ω and reaches a maximum for $\Omega \cong 0.023$. A second resonance occurs for $\Omega \cong 0.035$.

In Fig. 6, the amplitude initially increases with Ω and reaches a maximum for $\Omega \cong 0.013$. Other resonances occur for $\Omega \cong 0.020$, $\Omega \cong 0.025$, $\Omega \cong 0.029$, $\Omega \cong 0.033$, $\Omega \cong 0.036$, $\Omega \cong 0.039$, etc.

CONCLUSIONS

The hyperbolic heat conduction equation for a cylindrical solid with an internal heat generation due to an alternating current has been considered. The power generated per unit volume within the cylinder is nonuniform as a consequence of the skin effect. The equation has been written in a dimensionless form and has been solved analytically in a steady-periodic regime. It has been shown that the time-averaged dimensionless temperature distribution is independent of the value of the relaxation time τ, so that it coincides with the distribution obtained in ref. [14] according to Fourier's heat conduction, i.e. for $\tau = 0$. An analytical expression for the amplitude and the phase of thermal waves has been obtained. This expression has been employed to obtain plots of the amplitude of thermal waves at the surface of the cylinder as a function of the dimensionless parameter $\Omega = (\omega\mu\sigma)^{1/2}r_0$, for fixed values of the Biot number and of the dimensionless

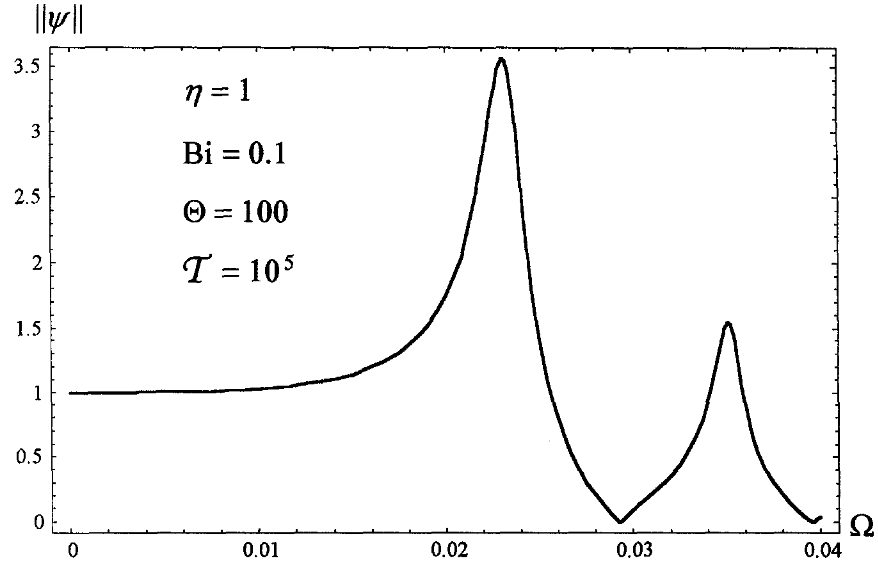

Fig. 5. Plot of the amplitude $\|\psi\|$ as a function of Ω for $\mathscr{T} = 10^5$.

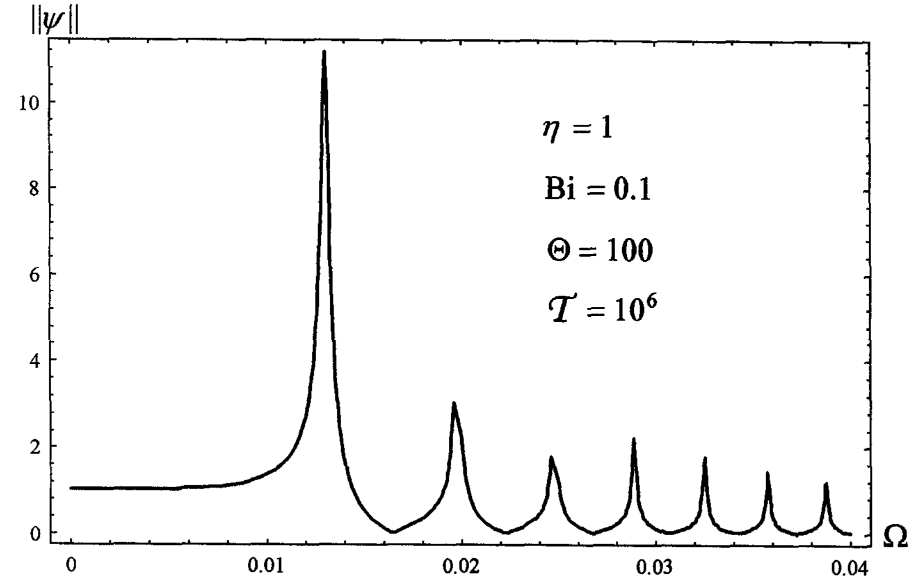

Fig. 6. Plot of the amplitude $\|\psi\|$ as a function of Ω for $\mathscr{T} = 10^6$.

parameters $\mathscr{T} = \tau/(\mu\sigma r_0^2)$ and $\Theta = 2/(\alpha\mu\sigma)$. It has been shown that the most important difference between the results obtained for Fourier's heat conduction (i.e. for $\mathscr{T} = 0$) and those obtained for hyperbolic heat conduction is the following: hyperbolic heat conduction determines the appearance of resonance phenomena. More precisely, for $\mathscr{T}$ equal to 10^3, 10^4, 10^5 and 10^6, local maxima of the amplitude of the dimensionless temperature oscillations as a function of Ω are observed, for fixed values of *Bi* and Θ. Moreover, for fixed *Bi* and Θ, the values of Ω which correspond to these maxima depend on $\mathscr{T}$, and the values of the amplitude which correspond to these maxima strongly increase for increasing values of $\mathscr{T}$. The significant dependence of the temperature distribution on the value of the relaxation time and the onset of thermal resonances suggest the possibility that the exact solution obtained in this paper may be employed as a parameter estimation method for experimental measurements of the relaxation time.

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